

Longitudinal Wave Transmission and Impact

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When a force is applied to a body, waves of stress and velocity radiate through the body from the point of application of the force. When these waves strike the boundaries of the body they may be wholly or partially transmitted to surrounding bodies or reflected back into the body, where they may reverberate back and forth within its confines, like sound waves within the body of air in a closed room. By superposition of these radiated and reflected waves the stress and motion of all parts of the body are gradually, and in general discontinuously, brought up to the values ordinarily assumed.

In the case of a small body this whole process takes place so quickly that there is great difficulty in detecting it, and generally it is of little practical importance. But there are cases where it is desirable to study the process in detail, particularly when the dimensions to be dealt with are no longer very small compared to the velocities of such waves, and in consequence the time element is no longer negligible, or where the time of application or variation of applied forces is very small and hence of the same order of magnitude as the aforementioned time element, as in cases of impact or harmonic forces of considerable frequency. In the present paper an attempt is made to consider some of the more important phases of this subject in as simple a manner as possible.

THE wave transmission considered here is "mechanical" wave transmission or the propagation of mechanical states of matter under the action of the simple laws of mechanics. The propagation of sound, of ripples on the surface of liquids, and of earthquake shocks are familiar examples. So-called "standing" waves, such as observed in the strings and pipes of musical instruments, can be considered to be the result of superposition of systems of transmitted waves.

When a force is applied to a given body, even one which we ordinarily think of as rigid, the body does not really behave as a rigid one—that is, it does not at once assume, as a whole, the state of stress or motion which we predict from elementary mechanics. When the force is applied, waves of stress and velocity radiate through the body from the point of application of the force.

When these waves strike the boundaries of the body, they may be wholly or partially transmitted to surrounding bodies (as sound waves to surrounding air, for instance) or reflected back into the body, where they may reverberate back and forth within the confines of the body, like sound waves within the body of air in a closed room. By superposition of these radiated and reflected waves the stress and motion of all parts of the body are gradually, and in general discontinuously, brought up to, or nearly up to, the values ordinarily assumed. It is necessary to say "nearly," because there is almost sure to be some loss of energy in this process, either to the surroundings

or through damping action within the body, although this loss is generally negligible.

In a small body this whole process takes place so quickly that we are unaware of it and would have great difficulty in detecting it, and in such cases it is generally of little practical importance.

But there are cases where it is desirable to study the process in detail, particularly when the dimensions to be dealt with are no longer very small compared to the velocities of such waves, and in consequence the time element is no longer negligible, or where the time of application or variation of applied forces is very small and hence of the same order of magnitude as the aforementioned time element, as in cases of impact or harmonic forces of considerable frequency. In the present paper an attempt is made to consider some of the more important phases of this subject in as simple a manner as possible.

LONGITUDINAL PRESSURE WAVES IN THIN,² PRISMATICAL BARS— GENERAL

Consider a long, thin uniform bar of elastic material, Fig. 1. If an impulse is applied to the end of this bar, say, by a uniformly distributed compressive force f acting on the end for a time t , then a "pressure wave" will be started at the end and will move along the rod. This "pressure wave" is merely a zone in which the material of the rod is under longitudinal compression.

In front of the zone and behind the zone the material is entirely uncompressed and at rest. At each instant new material in front of this zone, as the portion ab , is just about to be compressed, while at the same time material at the rear of the zone, cd , is losing its compression. Thus the zone can move along the rod with considerable velocity, although individual particles have only the small motion involved in compression and decompression as the zone passes.

Explanation. The explanation of this phenomenon is roughly as follows: When the force f is applied it compresses the material near the end of the rod. The compressed portion is shortened slightly, which means that some of it suffers a displacement, and in being displaced is given a velocity to the right. Due to its inertia it tends to continue moving at this velocity. It strikes the stationary material in front of it and compresses and starts this moving—but in so doing is stopped itself. Thus the zone of compression moves forward.

It should be noted that the material in the compressed zone always has a certain amount of velocity,³ while the material in front of and behind it is at rest. Hence we could equally well speak of this zone as a "velocity wave" or zone. We shall see that for a given bar there is a definite relation between the magnitude of the compression and this velocity of the particles; also both are respectively equal to the compression f and the velocity which f produced at the end of the bar when the wave was started.

The velocity of propagation of the wave or zone along the bar, on the other hand, is independent of f and is constant for a given material.

² The effect of lateral expansion, which would be of more importance in a thick bar, will be neglected.

³ This velocity of *particles* of material in the zone must not be confused with the velocity of *propagation* of the zone along the bar.

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NOTE: Statements and opinions advanced in papers are to be understood as individual expressions of their authors, and not those of the Society.

Quantitative Demonstration

Let f = magnitude of compressive force acting on bar
 t = time it acts
 L = length of pressure zone
 v = velocity of individual particles in the zone
 V = velocity with which the zone moves along the bar
 m = "inertia constant" of the bar, in this case the mass of the bar per unit length
 k = "stiffness constant" of the bar, in this case the modulus of elasticity of the material times the cross-sectional area of the bar.

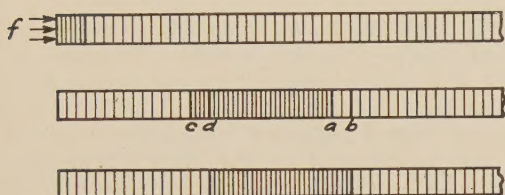


FIG. 1

Assume that, when the constant force f acts on the end of the bar for the time t , the result is a moving pressure zone of length L , all parts of which are under a uniform compression equal to f and have a uniform velocity v . The shortening of the bar due to the compression of this zone will be fL/k . The work done by the external force is therefore:

$$\frac{f^2 L}{k} \dots \dots \dots [1]$$

This must be equal to the energy stored in the pressure zone, which, no matter where the zone happens to be, consists of the

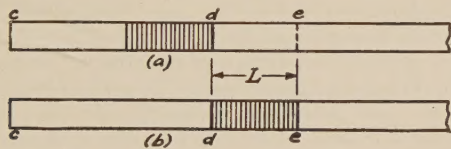


FIG. 2

potential strain energy $f^2 L/2k$ (assuming that the material obeys Hooke's law⁴) plus the kinetic energy $mLv^2/2$. Hence:

$$\frac{f^2 L}{k} = \frac{f^2 L}{2k} + \frac{mLv^2}{2} \dots \dots \dots [2]$$

from which

$$v = f \sqrt{\frac{1}{mk}} \dots \dots \dots [3]$$

that is, the velocity of particles in a wave equals the compression in the wave times a quantity which depends only on the bar. It is interesting to note that the energy in the zone is, from [2], half potential and half kinetic.

In Fig. 2 consider as a free body a definite portion of the bar de of length L . At a certain instant the pressure zone will be in the position shown in Fig. 2 (a), and, after an interval of time which we shall call for the present t' , it will have moved the distance L to the position shown in Fig. 2 (b). During this time t' the element de has had only one external force acting on it, namely, a force f acting on the end d to the right. The total change in momentum of the material in the element during this

time is $mL(v - 0) = mLf\sqrt{1/mk}$ (from [3]). The impulse of the external force must equal this change in momentum, or

$$ft' = mLf \sqrt{\frac{1}{mk}}$$

from which

$$\frac{L}{t'} = V = \sqrt{\frac{k}{m}} \dots \dots \dots [4]$$

that is, the velocity of propagation⁵ of the wave along the bar is $\sqrt{k/m}$, a constant depending only on the material of the bar as both k and m are proportional to the area of the bar. Also the length of the zone is

$$L = Vt' = Vt = \sqrt{\frac{k}{m}} t \dots \dots \dots [5]$$

as t' is obviously equal to t , the time the external force f acted on the end of the bar as the wave was started.

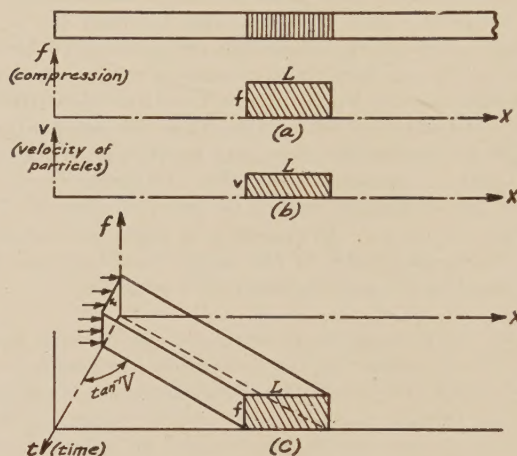


FIG. 3

We have still to prove our assumption that the velocity and compression are constant in all parts of the zone and that the compression equals f . This assumption is certainly true when the wave starts—that is, as it "crosses" the section c at the end of the bar, Fig. 2. Moreover, at the start at least, all longitudinal elements of the bar within the zone⁶ have on either side of them elements with the same velocity and pressure. Hence there is no tendency to change conditions within the zone either at this moment or later, and consequently the conformation of the zone remains the same as it started, throughout its motion along the rod (provided the rod is freely supported and no frictional resistances act).

The foregoing equations apply equally well if a tensile force acts on the end of the rod, starting a tension wave or zone in which the material is under tension. Tension may be considered to be a negative compression. Equations [3] and [4] show that when the sign of f is changed the sign of v is changed, but V remains the same as before. Thus in a compression wave the

⁵ This is the velocity of the wave relative to the bar. If the bar is moving, then the absolute velocity of the wave will be the vectorial sum of this velocity and the velocity of the bar. Thus sound has a greater absolute velocity with the wind than against it.

⁶ According to our assumption of a suddenly applied force, elements suffer an infinite acceleration when the edge of the pressure zone crosses them. However, the mass involved at any instant is infinitesimal, so it is possible for the unbalanced force f which acts on it at this instant to produce such an acceleration.

⁴ Pressure waves in materials which do not obey Hooke's law are discussed later on.

velocity of individual particles is in the same direction as the wave moves; but in a tension wave the velocity of the particles is in the opposite direction from that of the wave.

GRAPHICAL REPRESENTATION OF WAVES

The conditions in a wave such as has been discussed can be shown by diagrams, as in Fig. 3. Fig. 3(a) shows the compression f at all points in the rod at a given instant, ordinates above the x -axis representing compression and ordinates below representing tension. Fig. 3(b) shows the longitudinal velocity v of particles of the rod at a given instant, ordinates above the x -axis representing velocity to the right and ordinates below representing velocity to the left.

The conditions at all instants can be represented by a three-dimensional diagram such as shown in perspective in Fig. 3(c). In this diagram the conditions at any instant are given by a section parallel to the fx plane, as shown shaded.

METHODS FOR SIMPLIFYING COMPLEX PROBLEMS

If we establish the laws governing simple cases, such as the effect of a constant force under various conditions, it is possible in many cases to study more complex problems by replacing the complex action by an equivalent series of simpler actions and superposing their results.

When conditions are variable it is sometimes convenient to consider the variation to take place in several sudden steps;

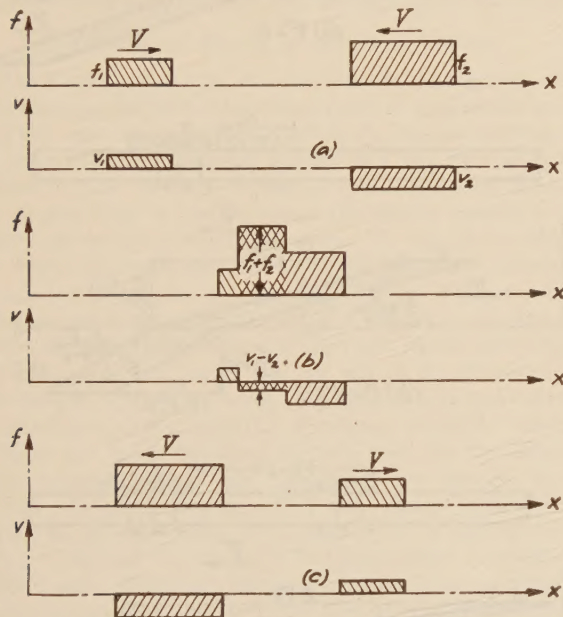


FIG. 4

it is then frequently obvious what the result would be if the number of steps were made infinite.

SUPERPOSITION OF WAVES

When two waves such as we have been considering come together, the principle of superposition holds as in other branches of mechanics, if there are no frictional resistances and the material follows Hooke's law. Hence the resulting compression or velocity is the vectorial sum (which in this case is the same as the algebraic sum) of the compressions or velocities in each wave.

When two waves coming from opposite directions meet, as

shown in Fig. 4, the compression in the resulting zone is $f = (f_1 + f_2)$, and the velocity is

$$v = (v_1 - v_2) = \frac{(f_1 - f_2)}{\sqrt{mk}} \dots \dots \dots [6]$$

It is evident that in this zone the relation between f and v given by [3] for a simple wave does not hold, and hence the energy is no longer half kinetic and half potential. However, the total energy in this zone is unchanged; it consists of the potential energy $(f_1 + f_2)^2 L/2k$ and the kinetic energy $mL(v_1 - v_2)^2/2 = (f_1 - f_2)^2 L/2k$ (from Equations [2] and [3]). The sum of these is $(f_1^2 + f_2^2)L/k$, which equals the energy before meeting. After

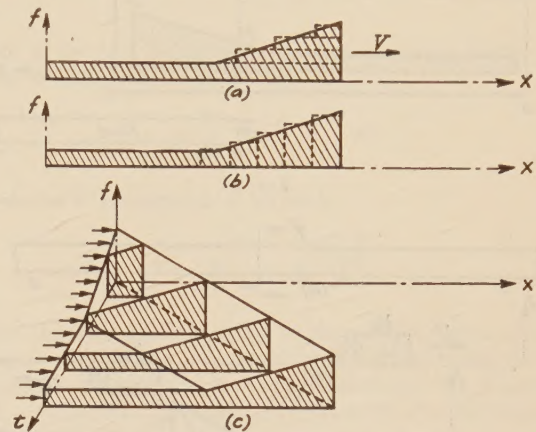


FIG. 5

passing, the waves return to their original conformation the same as if they had never met, as in Fig. 4(c).

If two waves going in the same direction superpose (as is possible when waves are started by forces at intermediate points along the rod) the resulting compression, by the principle of superposition, is $(f_1 + f_2)$ and the velocity $(v_1 + v_2)$. The relation between f and v in this case is the same as in a simple wave produced by a single force, Equation [3], and the energy will be half potential and half kinetic. However, the energy will be found to be considerably more than it would be if the superposed waves were separated. (This could also be predicted from the fact that, from [2], the energy of a wave varies as the square of f or v .) The explanation is that additional work is done by one external force due to the particle motion in the wave started by another external force.

WAVES PRODUCED BY VARIABLE FORCES

If the external force producing the wave varies—suppose, for example, it decreases uniformly for a while and then remains constant—we can imagine it to be replaced by a series of constant forces, either as indicated by Fig. 5 (a) or Fig. 5 (b). In this manner, if we know the laws governing waves produced by constant forces, we can apply them to cases where the forces vary.

It is very useful, for visualizing the action in complex cases, to note that as the wave moves along the rod it keeps the same conformation—just as though the fx curve were constructed on paper and drawn across the end of the rod and out along the rod. Thus in Fig. 6 imagine the shaded figure drawn on a piece of cardboard $abcd$ and the cardboard moved to the right past the fixed end of the rod e . The ordinate pq of the diagram passing the end of the rod represents the magnitude of the external force acting on the end of the rod at the instant. The fx diagram is thus seen to be a moving record of the value of the external force on the end of the rod; the ordinate rs , farthest

from the end, represents the initial value of this force, and, as we go back, the ordinates between give a complete history of the value of the external force up to the present. This is also shown by Fig. 5(c); the fx diagram at any instant is the same as the ft diagram up to that point (which is the history of the external force) except that the horizontal dimensions are multiplied by V .

WAVES DUE TO A FORCE APPLIED AT AN INTERMEDIATE SECTION

Consider a force f applied at section d , Fig. 7, for a time t . The effect is the same as if the force f acted partly on the end of

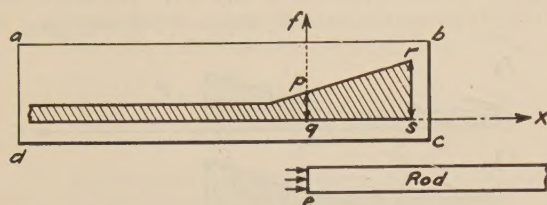


FIG. 6

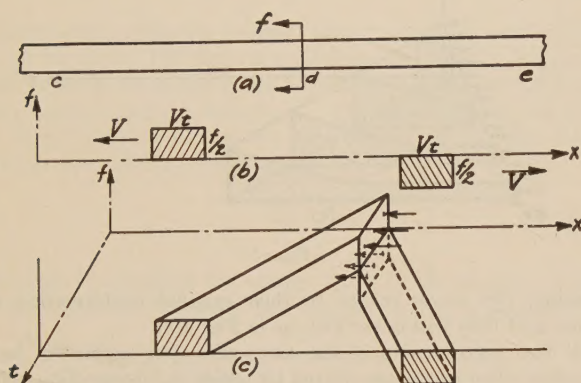


FIG. 7

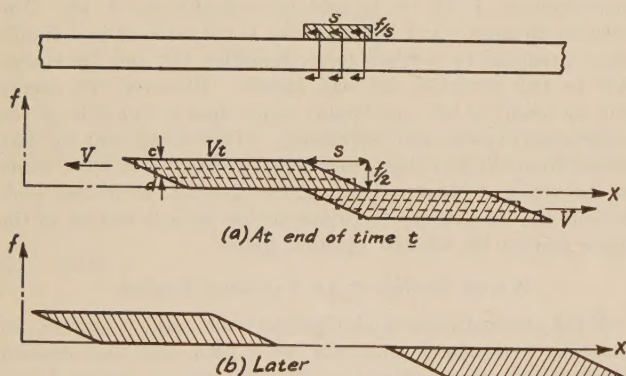


FIG. 8

a rod dc and partly on the end of a rod de , sending a compression wave of length $L = Vt$ along dc to the left, and a tension wave of the same length along de to the right. Since the velocity of particles at the section d is the same for both rods when the waves are started, the force f must, from [3], be equally divided between dc and de .⁷ Hence the wave traversing dc will be under

⁷ If two different rods are joined at d , the condition for equal velocities at d is, from [3]:

$$f_{cd} \sqrt{\frac{1}{m_{cd}k_{cd}}} = f_{de} \sqrt{\frac{1}{m_{de}k_{de}}}$$

and the force f would be divided between the two rods in the ratio given by this equation.

a compression of $f/2$ and the one traversing de under a tension of $f/2$, as shown in Fig. 7 [b] and [c].

If the force f is distributed, say, uniformly over a length s of the rod (that is, a distributed force f/s per unit length, acts for a time t), the result will be the same except that portions of the wave due to different parts of the force will be displaced lengthwise with respect to each other, as shown in Fig. 8. (In these figures the compression at any point is the vertical intercept of the shaded figures, as, for instance, the length cd .) The

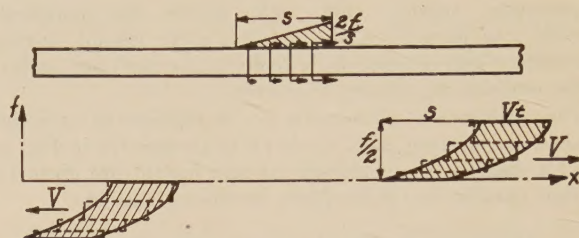


FIG. 9

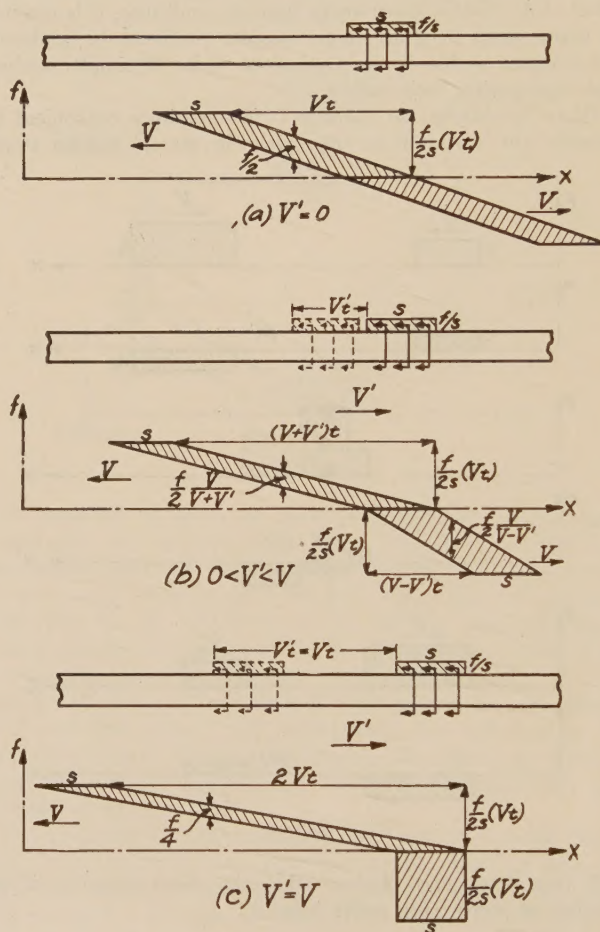


FIG. 10

broken lines show what the result would be if the distributed force were replaced by three concentrated forces; if it were replaced by a larger number of concentrated forces we should obviously approach the condition shown by the full lines.

Similarly a force with triangular distribution would give rise to the waves shown in Fig. 9. These waves, produced by distributed forces, have the same area of fx diagram as if the forces were concentrated. However, their energy is less, because the

energy depends on the square of f and the first power of L , from Eq. [1]; the explanation is that conditions are here the reverse of those considered before, under superposition of forces—there we were superposing the effects of forces, and here we are separating them.

WAVES PRODUCED BY TRAVELING FORCES

These are of interest in connection with the study of frictional resistances, longitudinal waves in railway trains, etc. Suppose that a force f is uniformly distributed over a length s and that its place of application moves uniformly along the rod (to the right in Fig. 10) at the rate V' . If the force is applied for a time t , then its place of application will move a distance $V't$ during this time.

When $V' = 0$ we have the case shown in Fig. 8(a); this can evidently be equally well represented in the form shown in Fig. 10 [a]. When V' is not zero we have the result shown in Fig. 10(b); this can be shown by replacing the moving force by a series of fixed forces acting successively at different points along the rod.

If $V' = V$ the conditions become as shown in Fig. 10(c). In this case the effect is cumulative on the wave going in the same direction as the forces. The compression in this wave, $(f/2s) \times (Vt)$, increases with time and is inversely proportional to s . Hence a concentrated force ($s = 0$) whose point of application moves with the velocity V would immediately produce infinite stresses—in practice failure or plasticity—and even a force not perfectly concentrated would soon produce such conditions.

LONGITUDINAL WAVES IN RAILWAY TRAINS

When a train passes over an unevenness in the track the effect is the same as that of a force whose point of application travels along the train at the speed at which the train is moving. A long train may be considered roughly as a bar with uniformly distributed mass (at least the part of the train behind the locomotive) and more or less uniformly distributed flexibility (due almost entirely to the coupling springs). It will have a definite rate of propagation of tension or compression waves, depending on the mass of the cars and the characteristics of the couplings.

The longitudinal external forces on a train are: (1) The frictional resistances to the motion of the train, which never change suddenly and so can hardly produce any of the effects we are considering. (2) Braking forces, which may be changed suddenly but which are usually uniformly distributed along the length of the train and hence balanced at each point by the resulting inertia forces, in which case they would not produce waves. (3) The pull of the engine, which may change suddenly and produce waves in the train of the same type as when a force is applied to the end of a bar. (4) The longitudinal component of gravity forces, which may act as a traveling force, as stated at the beginning. It should be noted that what we are interested in here is sudden *changes* in these forces, which are not immediately balanced locally as was assumed to be the case with braking forces. Thus if conditions are steady, with the engine exerting a pull of 5 tons, and its pull is suddenly increased to 8 tons, the effect is to superpose on the previous steady conditions a suddenly applied tensile force of 3 tons; if the pull is decreased the effect is that of a suddenly applied compressive force. If the change is spread over a considerable period of time the result is not so great, because by the time the full change is in effect there are likely to be other counterbalancing forces locally applied and the wave started by the first part of the change may have come back, reflected from the free end of the train as a wave of *opposite* sign, as considered later.

From the previous discussion it would seem that bad effects may be produced by the cumulative effect of gravity components

when the train traverses a change in the slope of the track at the same speed as the speed of propagation of longitudinal waves. Such effects might of course be controlled by designing the couplings so that the speed of propagation of waves would be above the speed range of the trains. Let b = length of each car, B = length of the train, W = weight of the train, and K = spring constant of a complete coupling (including the portion

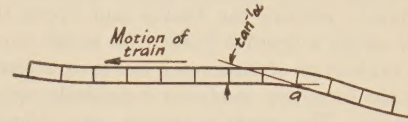


FIG. 11

on both of the cars connected). Then the average inertia and stiffness constants are respectively

$$m = \frac{W}{gB} \quad \text{and} \quad k = bK$$

The velocity of propagation of waves is

$$V = \sqrt{\frac{k}{m}} = \sqrt{\frac{bKgB}{W}}$$

When the train passes a *change* in slope of the tracks equal to α , as shown at a , Fig. 11, the change in longitudinal gravity component is a distributed force of approximately $W\alpha/B$ per unit length. If the train is moving to the left in Fig. 11, this is equivalent

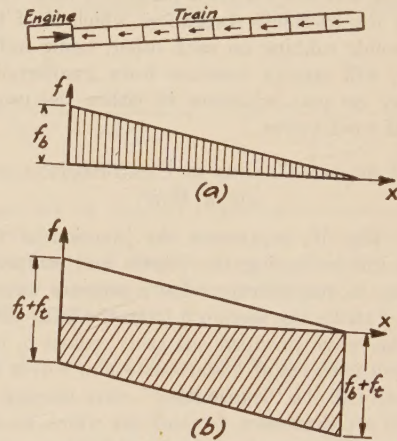


FIG. 12

lent to a traveling force acting to the left and moving along the train to the right a distance equal to the length of the train. Comparing this with Fig. 10(c) it is evident that the greatest effect is a tension in the rear of the train as it passes a , whose magnitude is $W\alpha B/2B = W\alpha/2$. If α is negative, this is a compression.

The greatest load on a coupling under steady conditions is likely to be at the front of a train going up a grade. The maximum tension would be $W'\alpha'$, where W' is the weight of the train minus the engine, and α' the absolute slope of the grade. The dynamic tension $W\alpha/2$ was calculated on the assumption that the train was homogeneous longitudinally. Actually it is a series of distinct units, and it is reasonable to suppose that as the wave front passes along such a series of units the tension will fluctuate above and below the average value $W\alpha/2$, so that the maximum tension may be greater than that calculated. The conditions in practice are more complicated than those assumed.

As another example consider the case of a railway train made up of cars having no individual brakes, so that any braking effort must be applied entirely at the engine. If such a train is descending a grade at a uniform speed, being held back by the braking force f_b applied by the engine (which will be equal to the longitudinal component of the weight of the train minus the frictional resistance), there will be a static state of compression along the train as indicated in Fig. 12(a). If now the engineer suddenly releases the brakes and opens the throttle sufficiently to apply a tractive force f_t (as is not unusual, when approaching the bottom of the grade), this is equivalent to superposing upon the foregoing condition a suddenly applied tensile force of $f_b + f_t$. This sends a tension wave along the train which, superposing on the previous condition of compression, produces the condition indicated in Fig. 12(b) when it reaches the end of the train. It will be observed that the coupling of the rear car is subjected to a tensile load of approximately $f_b + f_t$ at this instant, which may be sufficient to produce fracture.

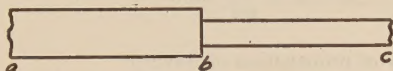


FIG. 13

In the propagation of pressure waves in trains the damping effect of friction is probably not as great as might be thought. If the velocity of the train is in the same direction as the velocity of particles v produced by a wave, or if it is in the opposite direction but is larger than v (as will usually be the case), then the total velocity of any part of the train will always be in the same direction, whether a wave is passing or not. Consequently that part of the frictional resistances which is of the type exhibited by solids rubbing on each other, being independent of the velocity, will exert a constant force unaffected by waves, and can play no part whatever in either the propagation or dissipation of such waves.

EFFECT OF A SUDDEN CHANGE IN CROSS-SECTION OR MATERIAL OF A BAR⁸

Section b , Fig. 13, represents the junction of two uniform portions ab and bc having the inertia and stiffness constants m_a, k_a and m_c, k_c , respectively. Let a pressure wave having the compression f strike the section b from the left. We may consider that this wave is wiped out as it crosses b , but that the pressure which it exerts at b starts two new waves from b —one which we shall call the “transmitted” wave moving to the right along bc with a compression f_t ; and one which we shall call the “reflected” wave moving back along ab with a compression f_r . This is shown in Fig. 14.

Equilibrium of an element of the rod at b during this process requires that⁹

$$f + f_r = f_t \dots \dots \dots [7]$$

The resultant velocity of particles to the left of section b during this process is, from [6],

$$v = \frac{f - f_r}{\sqrt{m_a k_a}}$$

and this must be equal to the velocity to the right of b , or

⁸ It will be assumed for the present that stresses are at all times uniformly distributed over the cross-sections, although this is of course not true locally when there is a change in the cross-sections. The effect of non-uniform distribution is discussed under “Gradual Changes.”

⁹ There are no changes of velocity *within* the waves considered, hence no inertia forces enter this equation of equilibrium.

$$\frac{f - f_r}{\sqrt{m_a k_a}} = \frac{f_t}{\sqrt{m_c k_c}} \dots \dots \dots [8]$$

Solving [7] and [8] for f_t and f_r , and letting $\sqrt{m_c k_c / m_a k_a} = r$, we find the compression in the transmitted wave is

$$f_t = \frac{2r}{r + 1} f \dots \dots \dots [9]$$

and the compression in the reflected wave is

$$f_r = \frac{r - 1}{r + 1} f \dots \dots \dots [10]$$

The velocity of particles in these waves can be found from [3]. The length of the reflected wave is the same as that of the original wave; the length of the transmitted wave is V_c/V_a or $\sqrt{m_a k_c / m_c k_a}$ times the length of the original wave. If the sudden change is one of area only, from A_a to A_c , then the lengths of all three waves are the same and $r = A_c/A_a$.

Equation [10] shows that if $r < 1$ (as when a wave strikes a reduction in area) the reflected wave is of opposite sign to the original—tension if the latter is compression, and compression if the latter is tension. If $r > 1$ (as when a wave strikes an enlargement) the reflected wave is of the same sign as the original. The transmitted wave is, from [9], always of the same sign as the original wave.

If $r = 1$ (that is, b is merely a section in a uniform bar) then [9] and [10] reduce to $f_t = f$ and $f_r = 0$, as expected.

FREE AND FIXED ENDS

If m_c and k_c are zero (that is, b is a free end of the bar) then r is zero, and we find from [9] and [10] that $f_t = 0$ and $f_r = -f$. This means that a wave is reflected back from a free end unchanged except for sign; a compression wave is reflected as a similar tension wave, and vice versa. Superposition of the incoming and outgoing waves at the free end makes the resultant compression there zero, while the resultant velocity of the particles at this point is

$$\frac{f - (-f)}{\sqrt{m_a k_a}} = \frac{2f}{\sqrt{m_a k_a}}$$

or twice the velocity in the original wave.

If k_c is infinite (that is, b is a rigidly fixed end) then r is infinite and we find $f_r = f$; hence the wave is reflected back entirely unchanged. Also $f_t = 2f$; however, the energy in this transmitted wave is, from [1], $f_t^2 L / k_c = 0$. Half of f_t can be considered as produced by the incoming wave, while the other half starts the outgoing wave. The resultant velocity of particles at this point is $(f - f_t) / \sqrt{m_a k_a} = 0$, as would be expected.

LONGITUDINAL IMPACT ON A PRISMATICAL BAR BY AN INFINITE MASS

If a perfectly rigid mass moving with a longitudinal velocity v_0 strikes the end of a bar squarely,¹⁰ a pressure wave will be started having a velocity of particles equal to v_0 , and hence a compression, from [3], of

$$f_0 = v_0 \sqrt{mk} \dots \dots \dots [11]$$

This force will also be acting on the mass, tending to decelerate it. If the mass is of infinite size, however, it will continue to

¹⁰ We must assume that contact takes place at the same instant over the whole surface of the end of the rod, although this condition is difficult to realize in practice.

move at the velocity v_o and the wave sent out will have a uniform compression f_o .

If the other end of the bar is free, this wave, on reaching the other end, will be reflected back with opposite sign. The portion coming back will superpose on the portion going out, producing a resultant compression $f_o - f_o = 0$ and a resultant velocity of particles $v_o + v_o = 2v_o$. When the reflected wave gets back to the starting point (which will be after a total time $2b/V$, where b is the length of the rod), the whole rod will be under zero stress and moving with a velocity twice that of the mass, so that it will separate from the mass at this instant and all waves will cease. This condition will be approximated if a large mass strikes a small free rod.

We also have a condition like this if two equal free rods c and d strike each other longitudinally. If the rods have initial velocities of v_c and v_d , respectively, the plane of contact will have a constant velocity of $(v_c + v_d)/2$ during the impact.¹¹ Hence the effect on each rod is the same as being struck by an infinite mass moving at this speed, $(v_c + v_d)/2$. Now for rod c ,

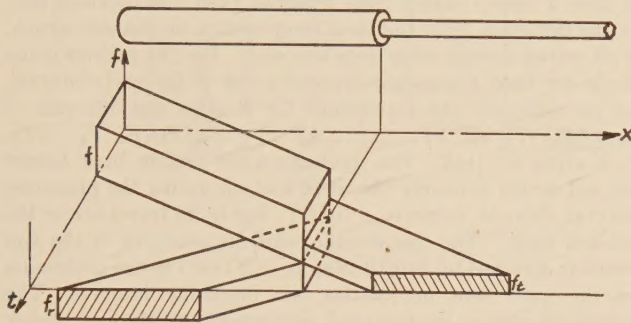


FIG. 14

this would be the same as if it were stationary and the infinite mass had a velocity of

$$\frac{v_c + v_d}{2} - v_c = \frac{v_d - v_c}{2},$$

within an isolated system having a uniform velocity of v_c . From the preceding paragraph the velocity of rebound inside this system will be twice the velocity of the infinite mass, or $2(v_d - v_c)/2 = v_d - v_c$. This, added to the velocity of the system, gives the absolute final velocity of c as $(v_d - v_c) + v_c = v_d$. Similarly, the final velocity of d will be v_c . The length of time the bars are in contact is of course $2b/V$, and the maximum compressive force is $\frac{v_d - v_c}{2} \sqrt{mk}$.

If the bars are of unequal length, b_c and b_d (suppose $b_c < b_d$) but are of the same materials and cross-sections, the conditions will be just the same as above at first, because the lengths can have no effect until the reflected waves get back. The pressure between the bars will become zero when the reflected wave returns in the shorter bar (after a time $2b_c/V$) and the conditions in the shorter bar c will be just the same as above; that is, all waves will have ceased and it will be moving at a velocity v_d . As the end of the longer bar in contact with it is now moving with the same velocity the bars will remain in contact, but without pressure, until the reflected wave returns in the longer bar (after a total time $2b_d/V$), and at this moment the longer bar will have a wave moving in it of total length $2b_c$ and a uniform

¹¹ If we consider the rods to be parts of an isolated system having a uniform velocity of $(v_c + v_d)/2$, then they will have symmetrical motion within this system, and from symmetry within this system their plane of contact must be stationary with respect to the system—and will thus have an absolute velocity of $(v_c + v_d)/2$.

compression $\frac{v_d - v_c}{2} \sqrt{mk}$. This wave will continue to move

along the rod after separation, being reflected from the two free ends with a change in sign at each reflection, until it is damped out by friction. From the law of conservation of momentum, after separation the total momentum of the longer bar divided by the mass of the bar (let us call this the "average" velocity of the bar, v_d') must remain constant and is found from

$$mb_c v_c + mb_d v_d = mb_c v_d + mb_d v_d'$$

which gives $v_d' = v_d + (b_c/b_d)(v_c - v_d)$. This is the velocity which d will have when the waves are all damped out; the kinetic energy in the system will then be $(m/2g)(b_c v_d'^2 + b_d v_d'^2)$,

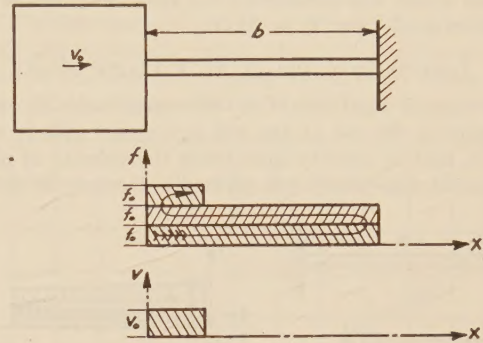


FIG. 15

while the original energy was $(m/2g)(b_c v_c^2 + b_d v_d^2)$. The ratio of lost energy to the original energy is therefore, using the above value of v_d' :

$$\frac{(b_c v_c^2 + b_d v_d^2) - (b_c v_d'^2 + b_d v_d'^2)}{(b_c v_c^2 + b_d v_d^2)} = \frac{\left(1 - \frac{b_c}{b_d}\right)(v_c - v_d)^2}{\left(v_c^2 + \frac{b_d}{b_c} v_d^2\right)}$$

If the bars are of unequal materials or cross-sections, and have the constants m_c, k_c , and m_d, k_d , then the plane of contact, during the impact, will have a constant velocity v_e of such a magnitude that the compressions of the waves started in the two bars will be the same, or:

$$f_c = (v_e - v_c) \sqrt{m_c k_c} = -f_d = (v_d - v_e) \sqrt{m_d k_d} \dots [12]$$

from which

$$v_e = \frac{\alpha v_c + v_d}{\alpha + 1}, \text{ where } \alpha = \sqrt{\frac{m_c k_c}{m_d k_d}} \dots [13]$$

As before, the pressure between the bars will drop to zero when the reflected wave returns in one of the bars, and actual separation will occur as soon as it returns in both of them, that is, after a time $2b_c/\sqrt{k_c/m_c}$ or $2b_d/\sqrt{k_d/m_d}$, whichever is larger. Following the same reasoning as before, if the reflected wave returned first in bar c , then this bar will be unstressed after separation and will have a velocity of $2v_e - v_c$. The bar d will have a wave moving in it, at separation, of length $2b \sqrt{k_d/m_d}/\sqrt{k_c/m_c}$ and with a uniform compression given by [12]. Using the law of conservation of momentum, the "average" velocity of this bar will be constant and equal to $v_d + 2\beta(v_e - v_c)$, where $\beta = m_c b_c/m_d b_d$. The maximum compression in each bar is given by [12], and the lost-energy ratio, obtained as before, is

$$\frac{4(\alpha - \beta)(v_c - v_d)^2}{(\alpha + 1)^2 v_c^2 + \frac{1}{\beta} v_d^2}$$

If a perfectly rigid infinite mass strikes one end of a bar which is fixed at the other end, the resulting wave (with compression given by [11]) will, on reaching the fixed end, be reflected with the same sign. The reflected portion, superposed on the portion going out, produces a resultant compression $f_o + f_o = 2f_o$ and a resultant velocity of particles $v_o - v_o = 0$. When the reflected wave gets back to the end struck, the velocity of this end must remain v_o ; hence the wave must be reflected again as from a fixed end, so as to make the resultant velocity due to the three superposed waves $v_o - v_o + v_o = v_o$; the resultant compression is $f_o + f_o + f_o = 3f_o$, as shown in Fig. 15. The wave will continue to emerge from the end struck and to be reflected from both ends without change of sign, so that the compression at the end struck will increase by the amount $2f_o$ at the end of every interval of time $T = 2b/V$.

LONGITUDINAL IMPACT BY A FINITE MASS.

If the mass is rigid but of a finite magnitude M , then the compression in the rod at the end in contact with it will decelerate it, and its velocity (and hence the velocity of particles in the rod at this point) will gradually decrease, as indicated

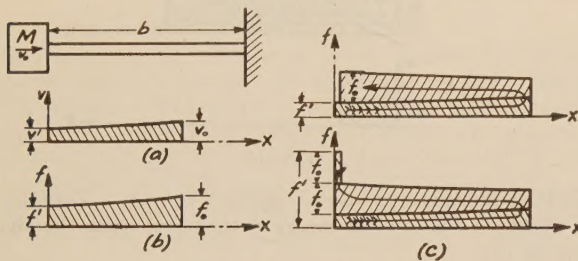


FIG. 16

by Fig. 16(a). Also, from [3], the compression in the rod at this point must decrease proportionally, as indicated by Fig. 16(b). Calling the compression and velocity of particles at the end struck f' and v' , the equation of equilibrium of the mass M is

$$f' + M \frac{dv'}{dt} = 0$$

From [3], $v' = \frac{1}{\sqrt{mk}} f'$, and this equation becomes, letting $\frac{\sqrt{mk}}{M} = B$:

$$\frac{df'}{f'} = -Bdt$$

Integrating,

$$\log_e f' = -Bt + c, \text{ or: } f' = C e^{-Bt}$$

Using the initial condition that $f' = f_o$ when $t = 0$, we find that $C = f_o$, hence

$$f' = f_o e^{-Bt} \dots \dots \dots [14]$$

After an interval of time $T = 2b/V$ (that is, just before the reflected wave returns), f' and v' will be

$$f' = f_o e^{-BT} = f_o e^{-2/\mu}, \quad v' = \frac{f'}{\sqrt{mk}} = \frac{f_o}{\sqrt{mk}} e^{-2/\mu} = v_o e^{-2/\mu}$$

where $\mu = M/m$ is the ratio of the mass of the striking body to the mass of the rod, so that

$$BT = \frac{\sqrt{mk}}{M} \frac{2b}{V} = \frac{\sqrt{mk}}{M} 2b \sqrt{\frac{m}{k}} = \frac{2mb}{M} = \frac{2}{\mu}$$

If the other end of the rod is free, the reflected wave will be of opposite sign and, when it returns, v' will be suddenly increased by v_o , the particle velocity of the wave front. As M will still be moving at the above velocity $v_o e^{-2/\mu}$, separation will occur at this instant. The "average velocity" of the rod after separation, found from the law of conservation of momentum, is $\mu v_o (1 - e^{-2/\mu})$. A wave of length $2b$ and a compression which can be found from [14] will be left moving in the rod. The maximum compression is of course f_o and the time of contact T . The lost-energy ratio, obtained as before, is:

$$1 - e^{-4/\mu} - \mu(1 - e^{-2/\mu})^2$$

If the other end of the rod is fixed the reflected wave will be of the same sign and f' will be suddenly increased by $2f_o$ at the end of every interval of time T , due to the return of the wave front, as shown in Fig. 16(c). We must therefore obtain a separate expression for f' for each one of these intervals. After the first interval there will be waves going out from the end struck and the same waves coming back to this end an interval of time T later, having been reflected from the opposite end. Let us designate by F the total compression, at the end struck, of all waves moving away from this end. For the reasons given above we need a separate expression for F for each interval. Let us designate the expressions for F after the intervals of time $0T, 1T, 2T, 3T, \dots, nT$ by: $F_0, F_1, F_2, F_3, \dots, F_n$. F_0 is given by [14]. The resultant wave coming back toward the end struck is merely the wave sent out during the preceding interval, delayed, however, a time T , due to its travel across the rod and back. The compression which it produces at the end struck is obtained by substituting $(t - T)$ for t in the expression for the wave sent out during the preceding interval. The general expression for the total compression at the end struck during any interval is therefore

$$f' = F_n + F_{n-1}(t - T) \dots \dots \dots [15]$$

where $F_{n-1}(t - T)$ means the expression F_{n-1} with t replaced by $(t - T)$. The net velocity of particles at the end struck is, from (6),

$$v' = \frac{1}{\sqrt{mk}} \left[F_n - F_{n-1}(t - T) \right] \dots \dots \dots [16]$$

The equation of equilibrium of the mass M during this interval is, as before:

$$f' + M \frac{dv'}{dt} = 0$$

Using [15] and [16] and $B = \sqrt{mk}/M$, this becomes

$$dF_n + BF_n dt = dF_{n-1}(t - T) - BF_{n-1}(t - T) dt$$

This can easily be integrated by multiplying through by e^{Bt} :

$$[e^{Bt} dF_n + BF_n e^{Bt} dt] = [e^{Bt} dF_{n-1}(t - T) + BF_{n-1}(t - T) e^{Bt} dt] - 2BF_{n-1}(t - T) e^{Bt} dt$$

$$d[e^{Bt} F_n] = d[e^{Bt} F_{n-1}(t - T)] - 2BF_{n-1}(t - T) e^{Bt} dt$$

Integrating and dividing by e^{Bt} ,

$$F_n = F_{n-1}(t - T) - 2Be^{-Bt} \int F_{n-1}(t - T) e^{Bt} dt + C \dots [17]$$

Using this we can find the expressions F_0, F_1, F_2 , etc., in turn starting with the known value of F_0 , the right-hand side of [17] being always given by the preceding expression which we have just determined. The integrals will be of known functions of t , which in our case are easily integrated. The integration constant C is found in each case by using the condition that

f' at the beginning of any interval is equal to $2f_0$ plus the value of f' at the end of the last interval, that is, when $t = nT$,

$$F_n + F_{n-1}(t - T) = 2f_0 + F_{n-1} + F_{n-2}(t - T)$$

In this manner we find that:

$$F_0 = f_0 e^{z_0}(1), \quad \text{where } z_0 = \frac{2}{\mu} \left(-\frac{t}{T} \right)$$

$$F_1 = F_0 + f_0 e^{z_1} (1 + 2z_1), \quad \text{where } z_1 = \frac{2}{\mu} \left(1 - \frac{t}{T} \right)$$

$$F_2 = F_1 + f_0 e^{z_2} (1 + 2 \cdot 2z_2 + 2z_2^2), \quad \text{where } z_2 = \frac{2}{\mu} \left(2 - \frac{t}{T} \right)$$

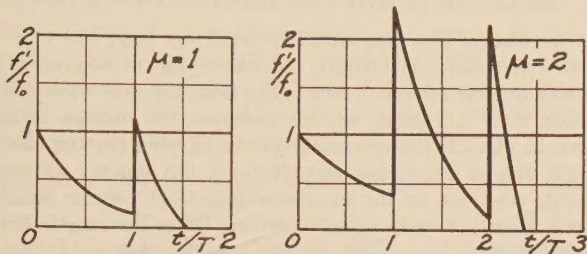


FIG. 17

$$F_3 = F_2 + f_0 e^{z_3} \left(1 + 2 \cdot 3z_3 + 2 \cdot 3z_3^2 + \frac{2 \cdot 2 \cdot 3}{3 \cdot 3} z_3^3 \right)$$

$$\text{where } z_3 = \frac{2}{\mu} \left(3 - \frac{t}{T} \right)$$

etc. The general expression for any interval is:

$$F_n = f_0 \sum_{m=0}^{n-1} e^{z_m} \sum_{h=0}^{m-1} \frac{m!(2z_m)^h}{(h!)^2(m-h)!} \quad \text{where } z_m = \frac{2}{\mu} \left(m - \frac{t}{T} \right)$$

.....[18]

and where h, m, n are integers, and $0! = 1$ (see footnote 12), $1! = 1$, $2! = 1 \cdot 2$, $3! = 1 \cdot 2 \cdot 3$, etc.

Using these expressions in [15] and [16], the values of f' and v' at all instants are determined for any value of μ . Fig. 17 shows the variation of f' with time, obtained in this way, for $\mu = 1$ and $\mu = 2$.

Separation occurs when f' becomes zero. The time of contact is therefore obtained by setting the various expressions for f' equal to zero and solving for t/T , using only such values of μ as give values of t/T within the interval for which each expression applies. In some cases, for a given value of μ , several values of t/T are obtained (in successive intervals) at which $f' = 0$. In such cases the smallest value of t/T only is used, because separation of course takes place the first time f' becomes zero. Subsequent values of t/T at which it becomes zero would have a meaning only if M were attached to the end of the rod. The results of such calculations are plotted in Fig. 18, which also shows an approximate expression for the time of contact, calculated empirically to match the actual curves as closely as possible and, when μ is very large, to approach the value obtained by neglecting the mass of the rod entirely (in which

case the end of the rod would undergo a half oscillation of simple harmonic motion):

$$\text{Time of contact} = \frac{b}{V} \left[\pi \sqrt{\mu + \frac{1}{2}} - \frac{1}{2} \right]$$

METHOD OF OBTAINING MAXIMUM COMPRESSION OR UNIT STRESS IN THE ROD DURING IMPACT

The maximum compression or unit stress in the rod during impact is obtained as follows: The total compression at any point in the rod during impact is always the sum of two values of F , one in the resultant wave going in one direction and one in the resultant wave going in the opposite direction. When the portion of wave with a compression equal to the maximum value of F strikes the fixed end and is reflected there, both waves will have this maximum value; the total compression at this point and at this instant is as great as can occur during the impact because at no other point or instant can we have two values

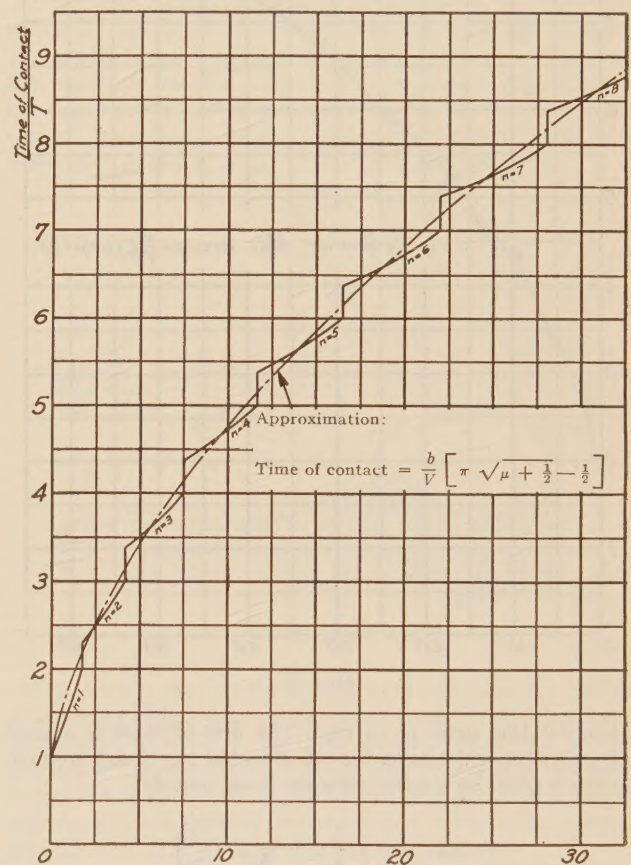


FIG. 18

of F greater than this.¹³ Hence the maximum compression during impact occurs in general at the fixed end and is equal to twice the maximum value of F . Now F varies in something the same fashion as f' , shown in Fig. 17, rising to a peak at the beginning of each interval and then decreasing. Hence in searching for the maximum value of F we need consider only the values at the beginning of each interval; the general expression for these is obtained by substituting $t/T = n$ in [18], which makes $z_m = 2/\mu(m - n)$. This gives us an expression for F/f_0 as a function of μ and n only. It is then only a question of what

¹² That this is not merely a convention can be shown as follows: $m! = \frac{(m+1)!}{m+1}$, and if $m = 0$, we have $0! = \frac{(0+1)!}{0+1} = 1$.

¹³ F may have two equal maximum values an interval of T apart which combine at one instant at the end struck.

value of n to use to get the greatest F for any particular value of μ . This is solved by plotting F/f_0 against μ for each value of n (which must be an integer) and selecting only those portions of the resulting curves which rise above their fellow-curves. This has been done in Fig. 19, except that $2F/f_0$ is plotted instead of F/f_0 , so that the resulting curves give the maximum compression in the rod for any value of μ .

The empirical curves $\text{Max. } f = f_0 (1 + \sqrt{\mu + 1})$ and $\text{Max. } f = f_0 (1 + \sqrt{\mu})$, calculated as above, practically form an en-

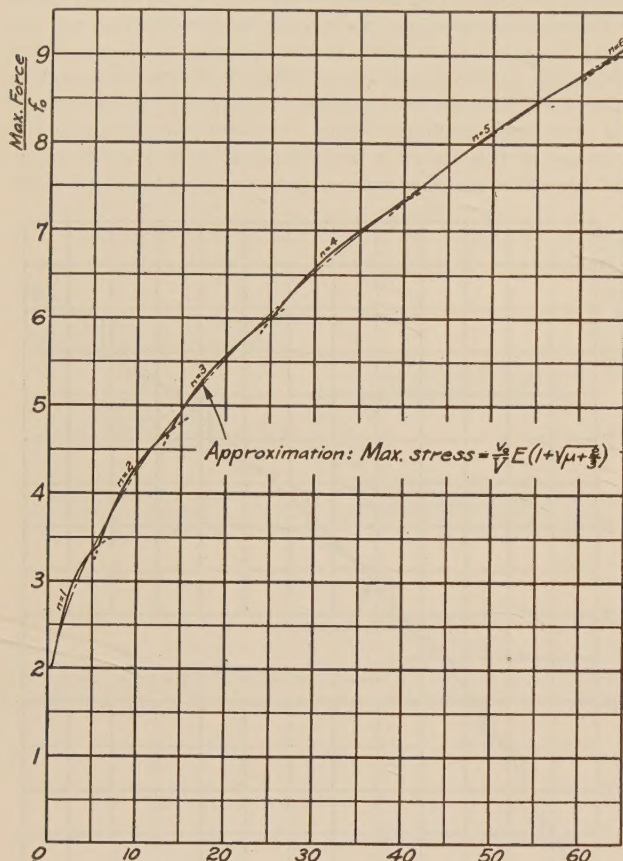


FIG. 19

velope for this series of curves. The first of these is a good approximation which is on the safe side, but the closest approximation is given by a curve between these, namely

$$\text{Max. } f = f_0 \left(1 + \sqrt{\mu + \frac{2}{3}} \right)$$

as shown on the chart.

The velocity of the mass M at separation (and from this the lost-energy ratio) can be obtained by taking values of μ , t/T , and n from Fig. 18 and substituting them in [18], and then in [16]. The results of such calculations are plotted on Fig. 20. A rough approximation to this very irregular function, calculated as above and also plotted on the chart, is:

$$\text{Velocity of } M \text{ at separation} = v_0 \sqrt{1 - \frac{2}{\mu^{1.3} + 3}}$$

With this approximation the lost-energy ratio, obtained as before, is $\frac{2}{\mu^{1.3} + 3}$.

EFFECT OF A GRADUAL CHANGE IN CROSS-SECTION OR MATERIAL OF BAR¹⁴

A gradual change in section or material may be considered to be brought about in a series of small, sudden steps such as previously considered. As a wave is transmitted through such a series of steps there will be reflection at each step. The reflected waves must travel back through the steps, and in so doing give rise to doubly reflected waves, and these to trebly reflected waves, and so on. The original transmitted wave combines with all waves due to an even number of reflections to form a resultant forward-moving wave, which we shall call the resultant transmitted wave f_t . The waves due to an odd number of reflections combine to form a backward-moving wave which we shall call the resultant reflected wave f_r .

SMALL CHANGE WITH COMPARATIVELY LONG WAVES

If the change is a comparatively small one (say, $4 > r > 1/4$, where r is defined as before), the waves due to successive reflections quickly approach zero. If a long, uniform wave passes through such a change, as, for instance, the change in area shown in Fig. 21, the conditions in the varying portion change at first due to the successive reflections, but quickly approach a stable condition as the successive reflections become smaller. After a length of wave equal to several times the length of the varying portion has passed, we can consider that conditions in this portion of the bar are constant with time. If they are constant with time there are no accelerations or inertia forces, and equilibrium requires that the total compression must be constant along the length of the varying portion. The total velocity of particles must also be constant along the length, as otherwise the compression would change with time. Consequently the compression of the resultant reflected wave f_r at the beginning of the varying portion and of the resultant

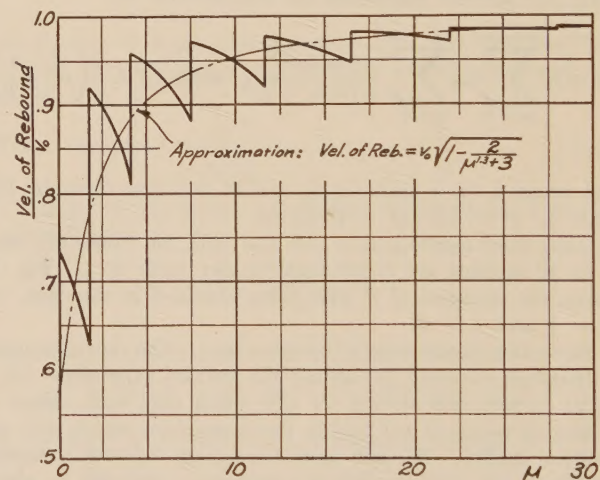


FIG. 20

transmitted wave f_t at the end of the varying portion are found by the same reasoning as for a sudden change, and are given by [9] and [10]. The magnitudes of f_r and f_t at any intermediate section such as b , Fig. 21, are found from the conditions that the resultant compression remains constant, or

$$f_t + f_r = f + \frac{r-1}{r+1} f$$

¹⁴ It is assumed that the waves are started on a plane perpendicular to the axis of the rod and remain so. This will not involve great error if the changes in section are gradual.

and that the resultant velocity remains constant, or

$$\frac{f_t - f_r}{\sqrt{m_b k_b}} = \frac{f - \frac{r-1}{r+1}f}{\sqrt{m_a k_a}}$$

where m_b and k_b are the constants at section b . Solving for

$$f_t \text{ and } f_r \text{ and letting } \sqrt{\frac{m_b k_b}{m_a k_a}} = r_b,$$

$$f_t = \frac{r + r_b}{r + 1} f \quad \text{and} \quad f_r = \frac{r - r_b}{r + 1} f$$

It is well known that when a bar with a *sudden* change in cross-section (such as shown by the broken lines in Fig. 21) is under compression, the material at the corners (shown shaded in the figure) is more or less dead material which takes little part in the action. It may be presumed, therefore, that in the case of a sudden change of cross-section, conditions are actually of the nature of those just described,¹⁵ instead of as previously obtained under the assumption that stresses were uniformly distributed over the cross-section at all points.

GRADUAL CHANGE WITH COMPARATIVELY SHORT WAVES

Let us consider that portion of the wave which is directly transmitted without any reflection, calling its compression f_t' . We can consider that the change is brought about in n successive steps, as suggested by the broken lines in Fig. 22, where n is to be made infinite. We can take these steps such that, letting $r = \sqrt{m_c k_c / m_a k_a}$ as before:

$$r^{1/n} = \sqrt{\frac{m_1 k_1}{m_a k_a}} = \sqrt{\frac{m_2 k_2}{m_1 k_1}} = \sqrt{\frac{m_3 k_3}{m_2 k_2}}, \text{ etc.}$$

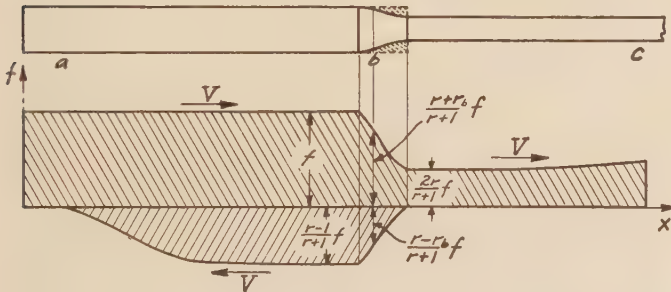


FIG. 21

If a wave with the compression f at section a travels to section c , the compression in the *directly* transmitted wave will, from [9], be multiplied by $2r^{1/n} / (r^{1/n} + 1)$ at each step, so that after passing all n steps, the final value of the directly transmitted wave is

$$f_t' = \left[\frac{2r^{1/n}}{r^{1/n} + 1} \right]^n f = \left[\frac{2}{2 + (r^{1/n} - 1)} \right]^n r f = \frac{r f}{[1 + \frac{1}{2}(r^{1/n} - 1)]^n}$$

When n becomes infinite, $(r^{1/n} - 1)$ becomes infinitesimal, so we can write $[1 + \frac{1}{2}(r^{1/n} - 1)]^n = [1 + (r^{1/n} - 1)]^{n/2} = [r^{1/n}]^{n/2} = \sqrt{r}$.

Hence:

$$f_t' = \sqrt{r} f \dots \dots \dots [19]$$

If the change is one of cross-section only, and the cross-sections are all geometrically similar, f_t' will therefore vary directly as

¹⁵ However, we are still making the approximate assumption that wave fronts are perpendicular to the axis of the rod.

the linear dimensions of the cross-section, as shown in Fig. 23. The intensity of stress, on the other hand, will vary *inversely* as the linear dimensions of the cross-section. A wave moving toward a pointed end would thus theoretically produce an infinite stress at the end. In a conical bar f_t' varies directly as the distance from the apex, and the unit stress inversely as this distance.

The reflected waves, whatever they are, obviously start from zero at the front of the transmitted wave, as suggested by the line gh , Fig. 23. Consequently the *resultant* transmitted wave f_t must have the value given by [19] at the front of the wave (as suggested by the broken lines in Fig. 23), and the resultant reflected wave must have the value zero at the front of the wave.

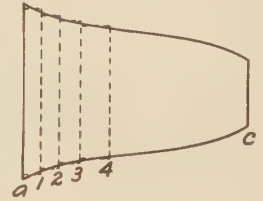


FIG. 22

If the wave is very short, such as the portion bc of the wave shown in Fig. 23, the resultant transmitted wave will be nearly as given by [19] throughout, as well as at the front.¹⁶ Moreover the resultant reflected wave will be of minor importance in this case, as can be shown by considering the energies in the waves. The energy in the resultant transmitted wave as it passes any section c , from [1] and [19], will be nearly

$$\frac{(\sqrt{r} f)^2 L_c}{k_c} = \frac{r f^2 L_c}{k_c} = \frac{\sqrt{\frac{m_c k_c}{m_a k_a}} f^2 \sqrt{\frac{k_c}{m_c}} t}{k_c} = \frac{f^2 \sqrt{\frac{k_a}{m_a}} t}{k_a} = \frac{f^2 L_a}{k_a}$$

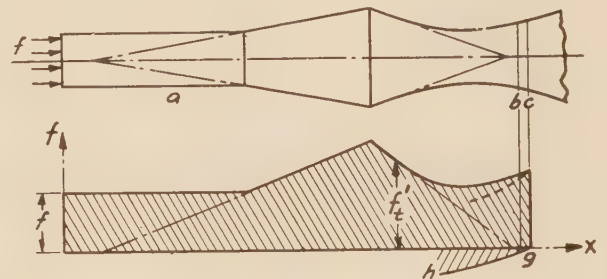


FIG. 23

which is the energy of the original wave starting at a . As the total energy cannot change, the energy in the resultant reflected wave must be small (it has been pointed out before that the energies of waves going in opposite directions can be calculated independently). It can be shown that the foregoing reasoning applies also to a series of short waves of *opposite* sign, such as are produced by a harmonic force of high frequency.

GENERAL CASE OF A GRADUAL CHANGE

To study the problem in more detail we must use the differential equation of the rod. Let $f = (f_t + f_r)$ represent the total compression at any section, u the longitudinal displacement of the section, and t the time. In general f varies both with time and with x . The rate of variation along the length (that is with x) is represented by $\partial f / \partial x$. The difference between the compressions at the two ends of an elemental length of the bar, of length dx , is therefore $(\partial f / \partial x) dx$. This represents an unbalanced force, acting on the element, which must equal the mass

¹⁶ It will be the same whether the portion bc is alone or at the front of a long wave, because the effect of the portion of the wave behind section b cannot catch up with the portion bc .

of the element times its acceleration $(mdx) \frac{\partial^2 u}{\partial t^2}$. Hence, taking the direction in which x is measured as positive,

$$-\frac{\partial f}{\partial x} dx = m dx \frac{\partial^2 u}{\partial t^2}, \text{ or } -\frac{\partial f}{\partial x} = m \frac{\partial^2 u}{\partial t^2} \dots [20]$$

The unit strain is

$$\frac{\partial u}{\partial x} = -\frac{f}{k} \dots [21]$$

The negative sign is due to the fact that we have taken f as compression. In these equations m and k are of course no longer constants but are functions of x . From [21],

$$f = -k \frac{\partial u}{\partial x}$$

Differentiating with respect to x ,

$$\frac{\partial f}{\partial x} = -k \frac{\partial^2 u}{\partial x^2} - \frac{dk}{dx} \cdot \frac{\partial u}{\partial x}$$

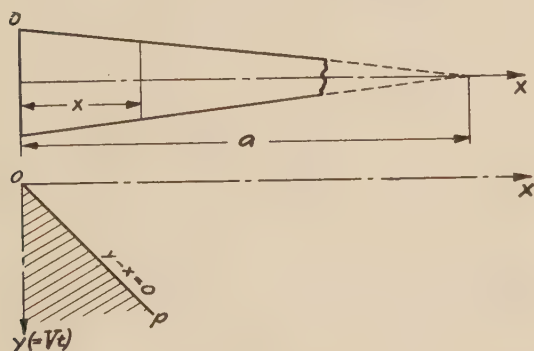


FIG. 24

and substituting this in [20] we have the general differential equation of the bar

$$k \frac{\partial^2 u}{\partial x^2} + \frac{dk}{dx} \cdot \frac{\partial u}{\partial x} = m \frac{\partial^2 u}{\partial t^2}$$

or

$$\frac{m}{k} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{k} \frac{dk}{dx} \cdot \frac{\partial u}{\partial x} \dots [22]$$

Also, from [21],

$$f_t + f_r = f = -k \frac{\partial u}{\partial x}$$

and using [6],

$$f_t - f_r = \sqrt{mk} v = \sqrt{mk} \frac{\partial u}{\partial t} \dots [23]$$

from which

$$\begin{aligned} f_t &= -\frac{k}{2} \left(\frac{\partial u}{\partial x} - \sqrt{\frac{m}{k}} \frac{\partial u}{\partial t} \right) \\ f_r &= -\frac{k}{2} \left(\frac{\partial u}{\partial x} + \sqrt{\frac{m}{k}} \frac{\partial u}{\partial t} \right) \end{aligned}$$

CONICAL BAR

As an example of the application of these equations, consider the case of a conical bar of homogeneous material, dimensioned

as shown in Fig. 24. If m_o and k_o are the constants at section o , then the constants at any other section will be¹⁷

$$m = \frac{(a-x)^2}{a^2} m_o = \left(1 - \frac{x}{a}\right)^2 m_o; \quad k = \left(1 - \frac{x}{a}\right)^2 k_o. [24]$$

In this case $V = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_o}{m_o}}$ is a constant. Using the notation $y = Vt$, Equation [22] becomes:

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} - \frac{2}{a-x} \frac{\partial u}{\partial x} \dots [25]$$

The general solution of this can be expressed as follows:

$$u = \frac{\phi(y-x) + \psi(y+x)}{1 - \frac{x}{a}} \dots [26]$$

where ϕ and ψ are any functions; they must of course be determined to fit the other conditions. Using Equations [23],

$$f = -k \frac{\partial u}{\partial x} = k_o \left[\left(1 - \frac{x}{a}\right) (\phi' - \psi') - \frac{1}{a} (\phi + \psi) \right] \dots [27]$$

$$v = \frac{\partial u}{\partial t} = V \frac{\partial u}{\partial y} = V \frac{\phi' + \psi'}{1 - \frac{x}{a}} \dots [28]$$

$$f_t = -\frac{k}{2} \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) = k_o \left[\left(1 - \frac{x}{a}\right) \phi' - \frac{1}{2a} (\phi + \psi) \right] \dots [29]$$

$$f_r = -\frac{k}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) = k_o \left[\left(\frac{x}{a} - 1\right) \psi' - \frac{1}{2a} (\phi + \psi) \right] \dots [30]$$

where $\phi' = \frac{d\phi}{d(y-x)}$, $\phi'' = \frac{d^2\phi}{d(y-x)^2}$, etc.

Suppose for example that a constant compressive force f_o is applied at o . In determining the value of u in the resulting wave an xy diagram, such as shown in Fig. 24, is useful. This is similar to the xt diagrams used previously but here $y = Vt$ represents the distance traveled by the wave. If we start measuring the time when the force is applied then the line $y = x$ represents the position of the wave front at any instant. Points in the area xop represent points along the rod before the wave reaches them, that is, beyond the wave front. The portion in which we are interested is the shaded area poy , which represents points within the wave. There are no discontinuities within this region and we can avoid discussion of discontinuities by making use of our knowledge, previously obtained, of the conditions at the wave front—that is, along the line op . We need only satisfy the known conditions along the lines op and oy , and the condition for variation between; the latter condition, represented by [25], has already been satisfied by using [26]. The condition along oy —that is, at the end of the bar—is that $(f)_{x=o} = f_o$. From [19] the condition along op , at the wave front, is that $(f)_{y=x} = \frac{a-x}{a} f_o = \left(1 - \frac{x}{a}\right) f_o$. It is found that ψ may be zero in this case, so that these conditions become

$$k_o \left[\phi'(y) - \frac{\phi(y)}{a} \right] = f_o \dots [31]$$

¹⁷ The following analysis applies to any case where [24] holds whether the bar is actually conical or not. It is assumed, as before, that cross-sections and the rate of change, or angle of taper, are small.

$$k_0 \left[\left(1 - \frac{x}{a} \right) \phi'(o) - \frac{\phi(o)}{a} \right] = \left(1 - \frac{x}{a} \right) f_0 \dots [32]$$

Solving the differential Equation [31], and using [32] to determine the resulting constant of integration,¹⁸

$$\phi = \frac{f_0 a}{k_0} \left(e^{\frac{y-x}{a}} - 1 \right), \psi = 0 \dots [33]$$

Substituting this in [24],

$$f = \left(1 - \frac{x}{a} e^{\frac{y-x}{a}} \right) f_0$$

If a is made very large we approach the condition in a uniform bar, $f = f_0$.

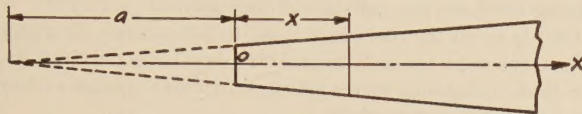


FIG. 25

If the force is removed after a length of time equal to L/V , this is equivalent to superimposing on the above wave a negative wave produced by a force $-f_0$, and having $(y-L)$ instead of y as the distance traveled by its wave front. This gives

$$(y-L) < x < y : f = \left(1 - \frac{x}{a} e^{\frac{y-x}{a}} \right) f_0, \text{ as above}$$

$$0 < x < (y-L) : f = \frac{x}{a} \left(e^{\frac{y-x-L}{a}} - e^{\frac{y-x}{a}} \right) f_0$$

The latter expression represents a sort of tail strung out behind the main part of the wave, due to reflections. All these waves are of course made up of two waves, one with compression f_1 moving forward, and one with compression f_2 moving backward. These can be investigated by substituting [33] in [29] and [30].

If a is negative we have the case shown in Fig. 25. The above formulas can be used for this case also, using a *negative* value for a .

WAVES IN MATERIALS WHICH DO NOT FOLLOW HOOKE'S LAW

Suppose that a uniform bar is made of a material such that, in a tensile or compressive test, we obtain a load vs. unit strain curve as shown in Fig. 26. The slope of this curve corresponds to our stiffness constant k . If a constant force f_1 is applied to the end of the bar, a wave with this compression will move along the bar with a velocity $V_1 = \sqrt{k_1/m}$. If now another force f_2 is superposed on f_1 , the bar, being already under a compression f_1 , will follow the load-strain curve with slope k_2 , and the wave started by f_2 will move along the bar with the velocity $V_2 = \sqrt{k_2/m}$. If the two forces are applied at the same time the action will be the same, and the complete wave front will present the appearance shown in Fig. 26(b), with the gap between the two parts of the front increasing with time.

If we imagine k_2 gradually decreased we see that when it becomes zero the second portion of the wave, having zero velocity, will never get started. This means that, if a force exceeding the yield-point force is applied to a bar of steel having a sharp, horizontal yield point, a wave will move along the rod with a compression equal to the yield-point force only. The work done by the remainder of the force is obviously absorbed as plastic work around the point of application.

¹⁸ It would ordinarily require two arbitrary constants to satisfy [32]; the fact that it can be done with only one is a check on our previous analysis of the conditions at the wave front.

If the load-strain curve of the material has a gradual decrease of slope with increasing stress (as in the case of cast iron, for example) it is evident, from the above reasoning, that the wave front will have a gradual backward slope, as in Fig. 27, increasing with time. Its exact shape will depend on the stress-strain diagram. The simplest result would be to have it a straight line, as indicated by the broken line in Fig. 27. In order for this to be the case the distance $\frac{t}{\sqrt{m}} (\sqrt{k_0} - \sqrt{k})$ in the figure would have to be proportional to f as well as to t , as for instance,

$$\frac{t}{\sqrt{m}} (\sqrt{k_0} - \sqrt{k}) = \frac{Ct}{\sqrt{mk_0}} f \dots [34]$$

where C is a constant to be determined. Substituting $k = df/de$ in this equation, and integrating and determining the constant of integration from the condition $f = 0, e = 0$, we find:

$$f = \frac{k_0 e}{C e + 1} \dots [35]$$

This equation represents the experimental results obtained with such materials as cast iron and annealed copper very well.

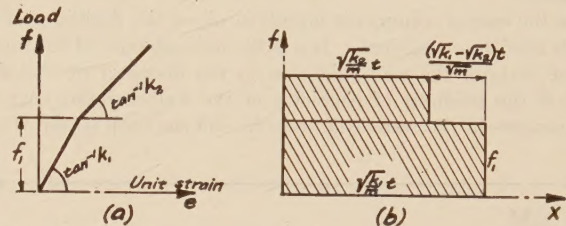


FIG. 26

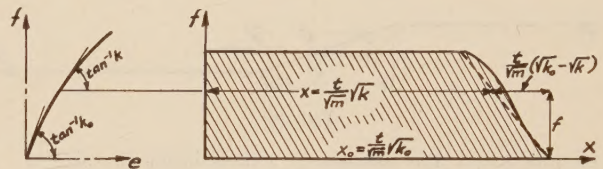


FIG. 27

The constants k_0 and C can be found from any two experimental values of f and e : f_1, e_1 , and f_2, e_2 (where f_1 is preferably the largest load used and f_2 about half the largest). Substituting these in [35] and solving for k_0 and C ,¹⁹

$$k_0 = \frac{f_1 f_2 (e_2 - e_1)}{e_1 e_2 (f_2 - f_1)}, \quad C = \frac{f_1 e_2 - f_2 e_1}{e_1 e_2 (f_2 - f_1)} \dots [36]$$

Hence we may assume that in bars of such materials the wave produced by a suddenly applied constant force will have, on the f, x diagram, an inclined front, the slope of which is $Ct/\sqrt{mk_0}$ (from [34]), where k_0 and C are given by [36]. If, upon unloading, the material shows a straight load-strain line having the slope k_0 —as seems to be approximately the rule—the rear of the wave will evidently be vertical in the f, x diagram and will move at the velocity $\sqrt{k_0/m}$, so that the total length of the wave will remain constant.

COMPRESSION WAVES IN HELICAL SPRINGS AND COLUMNS OF LIQUIDS

All the foregoing theory applies without change to longitudinal

¹⁹ k_0 should be the slope of the experimental f, e curve at the bottom but it is difficult to measure this accurately.

waves in helical springs. The stiffness constant k in this case is the spring constant of a unit length of the spring.

That portion of the foregoing which deals with prismatical bars can be applied to longitudinal pressure waves in inclosed columns of gases and liquids, of uniform section. It also applies approximately to the longitudinal propagation of surface waves in liquids in long uniform tanks; in this case, however, the lateral motions involved at the ends of the waves produce a large frictional effect which modifies conditions at these points.

In the case of columns of gases in rigid pipes, k is the area of the cross-section times the modulus of volume elasticity. The latter

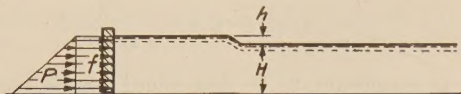


FIG. 28

for adiabatic conditions, is λp , where λ is the ratio of the specific heats of the gas under constant pressure and under constant volume ($\lambda = 1.4$ for air) and p is the unit pressure of the gas. For very long waves there will be some deviation from adiabatic conditions in the middle of the wave, which will complicate the problem.

In the case of columns of liquids in pipes, the flexibility of the walls must be considered. If p is the unit pressure of the liquid, α the ratio of the wall thickness to the diameter of the pipe, and E the modulus of elasticity of the wall material, then the hoop stress in the walls will be $p/2\alpha$ and the hoop strain $p/2\alpha E$.

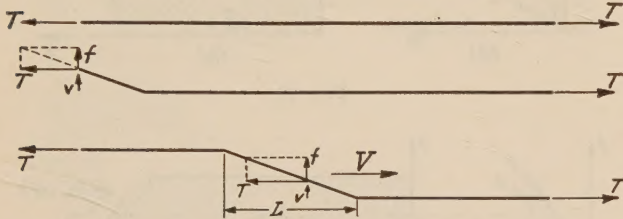


FIG. 29

This produces a unit increase in volume inside the pipe of

$$\left(1 + \frac{p}{2\alpha E}\right)^2 - 1 = \frac{p}{\alpha E}$$

approximately. This has the same effect as a unit decrease in volume of the liquid. This decrease must be added to that due to the compressibility of the liquid, which is p/E_v , where E_v is the modulus of volume elasticity of the liquid. The stiffness constant k , being the longitudinal force per unit longitudinal shortening, will be equal to the area of the cross-section times

$$\frac{p}{\frac{p}{E_v} + \frac{p}{\alpha E}} = \frac{E_v}{1 + \frac{E_v}{\alpha E}}$$

The inertia of the pipe walls and the stiffening effect of the portion of the pipe just outside the wave will modify conditions near the ends of the waves and will be of importance if the waves are very short.

The case of surface waves in long tanks is illustrated by Fig. 28. If a uniformly distributed longitudinal force f (superposed on the total static pressure, P) is exerted on the water at a certain point, say through a movable bulkhead as shown in the figure, the water will rise a distance h , such that

$$\frac{(h + H)^2}{H^2} = \frac{P + f}{P}$$

where H is the original depth. Assuming that the volume of water remains the same, there will be a unit shortening in the lengthwise direction, due to this increase in depth, of h/H . And k , the force per unit shortening, will be $f/(h/H)$. Using the above equation this reduces to $k = 2P$, approximately (if h/H is small).

APPLICATIONS TO OTHER TYPES OF WAVES

The foregoing theory can be applied also to the case of small lateral waves in a stretched chain or flexible rope. In this case f is to be taken as a lateral force acting on the rope, superposed on the initial constant tensile force T , and v is to be taken as the resulting lateral velocity of particles of the rope, as indicated in Fig. 29. The lateral displacement per unit length of rope can be taken as f/T , if f is small compared to T . In this case k , being the force per unit lateral displacement, is $f/(f/T) = T$. In this, as in all the preceding cases, m is the same as originally defined, the mass per unit length. It may be noted, in this last case, that superposing waves may be in different planes and must then be combined vectorially.

All the preceding theory can be applied to the case of torsion waves in bars of uniform or variable section (with even more exactitude than for the original case of longitudinal pressure waves, as there is no lateral expansion here to affect the conditions at the ends of waves). In this case f is to be interpreted as the torque applied to the bar (or the internal torque on sections within the wave) and v is to be interpreted as the resulting angular velocity of sections of the bar. For this case m must be taken as the moment of inertia of mass of the bar (about the longitudinal axis) per unit length of the bar; and k will be the torque required to twist unit length of the bar one radian, which can be determined by the usual methods.

Discussion

THOS. C. RATHBONE.²⁰ Professor Donnell has opened up a most fascinating subject which has practical application in many fields of mechanical engineering. The transmission of impulses in materials is usually associated with the case of relatively small parts involving extremely short periods of time. When these studies are applied to large central-station turbines, however, both the time intervals and distances become appreciable. For example, the distance between the bearings of a large generator rotor may reach 35 ft. or more, and a length of 75 ft. would not be unusual between the turbine thrust bearing and the outboard bearing. These units are normally mounted on a continuous steel foundation. Although the forces at the bearing due to unbalance in the rotor are usually considered as acting in a plane normal to the turbine axis, it is not unusual to find axial disturbances.

In his investigation of the vibration phenomena, the writer has made use of stroboscopic vibrometers to explore in some detail the characteristic of vibration of large installations as a whole, with particular regard to the phase relation between the vibratory motion and references on the rotor. These studies were directed toward arriving at a method of correcting unbalance on a complicated structure where the vibratory disturbances at the several bearings are transmitted throughout the unit and affect each other. On such an installation the motion observed at any bearing represents a composite which includes disturbances emanating from the other bearings.

Many unusual conditions were disclosed. For example, it was found that the same unbalance in a rotor would produce a

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given set of vibratory characteristics when rotating in the opposite direction at the same speed. It was also found that the so-called "angle of lag," between a point in the vibratory motion of the bearing and a datum on the rotating spindle, would vary through two full circles or more in going through a speed range up to 2000 r.p.m. These results are entirely at variance with the commonly accepted notion that the "angle of lag" at the critical speed is 90 deg., becoming 180 deg. at an infinite speed.

On these very large units the phase relation of the vibratory motion at the various bearings is affected by the length of time required for an impulse to travel from one point to another. For example, on an installation of the size mentioned, running at 1800 r.p.m., the spindle will turn through approximately 50 deg. during the time required for an impulse to travel from the

thrust bearing, for example, to the outboard bearing. Such large angles are of course very appreciable when attempting to analyze the vibratory characteristics in order to determine the proper locations for balance corrector weights.

The velocity of propagation in steel is about one-third higher than in concrete. It is interesting to note that a disturbance at the thrust end of such a unit would reach the outboard end through the steel and iron turbine structure about 17 deg. ahead of the wave coming through the concrete foundation.

The study of resonances in a large turbine-generator installation from the viewpoint of reflections and standing waves offers an interesting field of investigation. The modes of vibration commonly encountered on such structures are very complex and cannot be predicted by ordinary methods.
